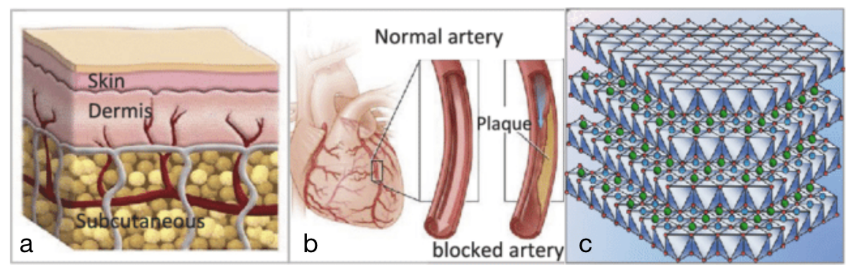
TRCQ for Viscoelastic Waves

Hasan Eruslu and Francisco-Javier Sayas   Department of Mathematical Sciences, University of Delaware

Introduction: Viscoelasticity

Viscoelasticity have been used to model deformation and stress in certain solids, including metals, polymers, and biological materials, when they are under external forces. **Example.** Some viscoelastic solids (See [6]):



a. Skin
b. Heart's artery wall
c. Polymers

Introduction: TRCQ

Trapezoidal rule based Convolution Quadrature method (TRCQ) [3,5] is a time discretization scheme which approximates a convolution equation:

$$u(t) = (f * g)(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau.$$

TRCQ approximation with time-step κ will be denoted as $u^\kappa = f^\kappa * g$.

Notation

What is given? We consider a viscoelastic material identified by:

- Mass density ρ,
- Viscoelastic material tensor C.

We work on its reference configuration with:

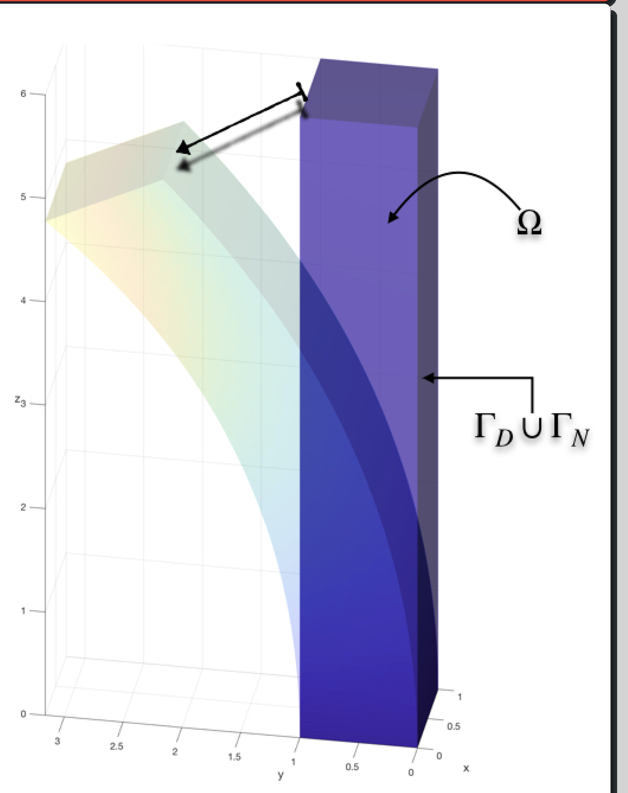
- Domain $\Omega \subseteq \mathbb{R}^d$,
- Boundary parts (Dirichlet and Neumann) Γ_D, Γ_N.

This solid deforms under:

- Forcing term $\mathbf{f}$,
- Dirichlet and Neumann data α, β.

We want to compute

- Displacement $\mathbf{u}$,
- Stress $\boldsymbol{\sigma}$.



Viscoelastic law

Defines a relation between stress and strain.

- **Stress**: $\boldsymbol{\sigma}(t) = \int_0^t \mathcal{D}(t-\tau)\varepsilon(\dot{\mathbf{u}}(\tau))\,\mathrm{d}\tau$,   **Strain**: $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$

Viscoelastic material tensor. Determines the Laplace domain relation:

$$\mathrm{C}(s) = s\mathcal{L}\{\mathcal{D}\}(s) \quad \forall s \in \mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}s > 0\}.$$

- This tensor is symmetric, positive and bounded such that
- $\|\mathrm{C}(s)\| \leq |s|^r \phi(\mathrm{Re}s) \quad \forall s \in \mathbb{C}_+$. Here $r \geq 0$, and $\phi(x^{-1})$ polynomially bounded.

Model problem and previous work

Viscoelastic wave. Consider the Sobolev spaces $\mathbf{H}^1(\Omega)$ and $\mathbb{H}(\mathrm{div}, \Omega)$.
Find (displacement) $\mathbf{u} : [0, \infty) \to \mathbf{H}^1(\Omega)$ and (stress) $\boldsymbol{\sigma} : [0, \infty) \to \mathbb{H}(\mathrm{div}, \Omega)$ such that for all $t \geq 0$

$$\begin{aligned} &\text{(Conserv. of Momentum)} && \rho\ddot{\mathbf{u}}(t) = \mathrm{div}\,\boldsymbol{\sigma}(t) + \mathbf{f}(t), \\ &\text{(Boundary Condition)} && \gamma_D \mathbf{u}(t) = \boldsymbol{\alpha}(t), \quad \gamma_N \boldsymbol{\sigma}(t) = \boldsymbol{\beta}(t), \\ &\text{(Initial Condition)} && \mathbf{u}(0) = \mathbf{0}, \quad \dot{\mathbf{u}}(0) = \mathbf{0}. \end{aligned}$$

Previous work. Detailed analysis of this PDE and properties of the viscoelastic tensor is done in our team paper last year [2]. FEM semi-discretization results were presented in WRS 2018.
Current work. Stability and error estimates of the fully-discretized system based on TRCQ.

Procedure

We approach the problem in the following way

- Consider the FEM semi-discretization: For a FEM space $\mathbf{V}_h \subseteq \mathbf{H}^1(\Omega)$, find $\mathbf{u}_h : [0, \infty) \to \mathbf{V}_h$
- Analyze the Laplace domain transfer function: $\mathrm{J}(s)(\mathbf{A}_h, \mathbf{B}, \mathbf{F}) = \mathbf{U}_h$
- Introduce TRCQ approximation: $\mathbf{u}_h^\kappa$ approximates $\mathcal{J} * (\boldsymbol{\alpha}_h, \boldsymbol{\beta}, \mathbf{f}) = \mathbf{u}_h$
- Study the fully-discretized system (TRCQ is a second order system [3])

Transfer function analysis

This function maps the data to the solution. We study the transfer function of FEM semi-discretization of the problem in Laplace domain.
Data. Force and Neumann data are same. We consider an approximation of Dirichlet data: $\mathbf{A}_h$.
Definition. For $s \in \mathbb{C}_+$, transfer function $\mathrm{J}(s)$ maps $(\mathbf{A}_h, \mathbf{B}, \mathbf{F})$ to $\mathbf{U}_h$ such that:

$$\gamma_D \mathbf{U}_h = \mathbf{A}_h, \quad s^2(\rho \mathbf{U}_h, \mathbf{w}_h)_\Omega + (\mathrm{C}(s)\varepsilon(\mathbf{U}_h), \varepsilon(\mathbf{w}_h))_\Omega = (\mathbf{F}, \mathbf{w}_h)_\Omega + \langle \mathbf{B}, \gamma \mathbf{w}_h \rangle_{\Gamma_N} \qquad \forall \mathbf{w}_h \in \mathbf{V}^h \cap \ker\gamma_D.$$

Stability analysis. This enables us to use TRCQ method.

Main result

Regularity of data. We assume that our data is sufficiently regular:

$$\mathbf{f} \in W_+^5(\mathbb{R}; \mathbf{L}^2(\Omega)), \quad \boldsymbol{\alpha}_h \in W_+^{2r+8}(\mathbb{R}; \mathbf{H}^{1/2}(\Gamma_D)), \quad \boldsymbol{\beta} \in W_+^6(\mathbb{R}; \mathbf{H}^{-1/2}(\Gamma_N)),$$

where $W_+^m(\mathbb{R}; X) := \{f \in \mathcal{C}^{m-1}(\mathbb{R}; X) \,:\, f^{(m)} \in L^1(\mathbb{R}; X),\ f \equiv 0 \text{ on } (-\infty, 0)\}$ are Sobolev spaces.
Theorem (TRCQ Approximation). For each time step $t_n = n\kappa$, $n = 0, 1, \ldots, N$ we have

$$\|\mathbf{u}_h(t_n) - \mathbf{u}_h^\kappa(t_n)\|_{1,\Omega} \leq \kappa^2 \Big(D_f(t_n) \int_0^{t_n} \|\mathbf{f}^{(4)}(\tau)\|_\Omega \mathrm{d}\tau \;+\; D_\alpha(t_n) \sum_{k=0}^{r+2} \int_0^{t_n} \|\, \boldsymbol{\alpha}_h^{(r+6+k)}(\tau)\|_{1/2,\Gamma_D} \mathrm{d}\tau$$
$$+\, D_\beta(t_n) \sum_{k=0}^{1} \int_0^{t_n} \|\boldsymbol{\beta}_h^{(5+k)}(\tau)\|_{-1/2,\Gamma_N} \mathrm{d}\tau \Big).$$

Notes

- Each piece of data produces different amount of error. (Details by transfer function analysis)
- Most expensive error: Dirichlet data. Lifting Dirichlet data requires a bound on $\mathrm{C}(s)$.
- Exact error scales are studied and they are all polynomials in time: D_f, D_α, D_β.
- For the models studied in [4]: $D_f(t), D_\alpha(t), D_\beta(t) \lesssim t^5$. (This covers the examples in this poster)

Key tools and contributions

Banjai's result. Our proof uses ideas from [1], improves them and shows that the error is not exponential in time.

Lubich's result. We also make use of the following result from [4]. This is a special case when $-1 < r \leq 0$. If $g \in W^{r+5}(\mathbb{R}; X)$ and f_κ is the TRCQ approximation of $f : [0, \infty) \to \mathcal{B}(X, Y)$, then for $t > 0$:

$$\|(f_\kappa - f) * g(t)\|_Y \leq C(t)\kappa^2 \int_0^t \|g^{r+5}\|_X \mathrm{d}\tau.$$

Here C is non-decreasing and polynomially bounded. Our contribution is to obtain the exact form of this function, and include the case $r = 0$.

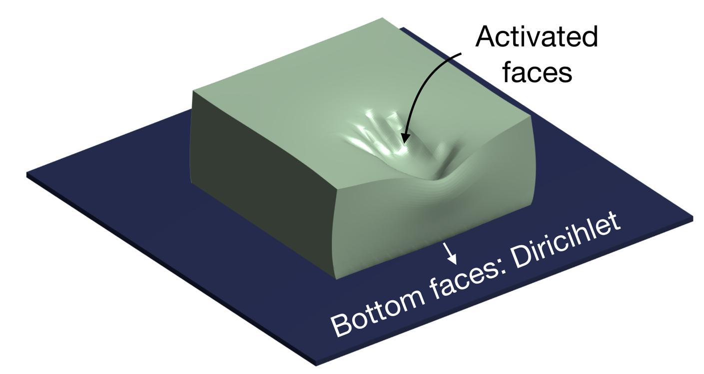
Examples and simulation design

Zener's fractional model. For symmetric tensors $\mathrm{C}_0, \mathrm{C}_1 > c > 0$ a.e., and fractional derivative of type Caputo with $\nu \in (0, 1)$:

- Viscoelastic material tensor: $\mathrm{C}(s) = (1 + as^\nu)^{-1}(\mathrm{C}_0 + s^\nu \mathrm{C}_1)$,
- Stress-strain relation (Time domain): $\boldsymbol{\sigma}(t) + a\partial^\nu \boldsymbol{\sigma}(t) = \mathrm{C}_0 \varepsilon(\mathbf{u}(t)) + \mathrm{C}_1 \varepsilon(\partial^\nu \mathbf{u}(t))$.

Isotropic model. A special case: $\mathrm{C}(s)\mathrm{M} = 2\mu(s)(\frac{1}{2}(\mathrm{M} + \mathrm{M}^\top)) + \lambda(s)(\mathrm{tr}M)\mathrm{I}$.

Desiging a simulation. In the experiments, we capture the memory effect of a light viscoelastic material ($\rho = 0.01$) with isotropic fractional Zener model of order $\nu = 0.7$. We impose the pressing-hand effect as Neumann boundary condition. Corresponding faces are activated during the hand-press time and then freed.



Future work

- Implementation on non-polyhedral domains, such as cylindrical, spherical.
- Using TRCQ approximation on poroviscoelastic materials. This involves coupling pressure diffusion coming from fluid flow in porous media with the deformation of the solid.

References

[1] L. Banjai. *Multistep and multistage convolution quadrature for the wave equation: Algorithms and experiments.* SIAM J. Sci. Comput., 2009.

[2] T. Brown, S. Du, H. Eruslu, and F.-J. Sayas. *Analysis of models for viscoelastic wave propagation.* Applied Mathematics and Nonlinear Sciences, 2018.

[3] C. Lubich. *Convolution quadrature and discretized operational calculus. I.* Numer. Math., 52(2): 129-145, 1988.

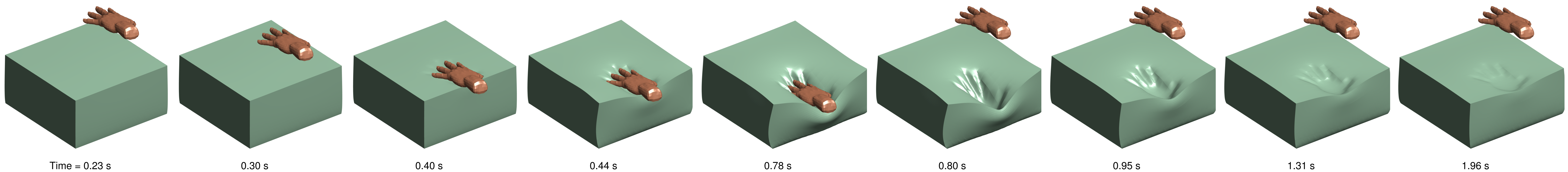
[4] C. Lubich. *On the multiple time-step discretization of linear initial-boundary problems and their integral equations.* Numer. Math., 67(3):365-389,1994.

[5] M. Hassell and F.-J. Sayas. Convolution quadrature for wave simulations. In *Numerical simulation in physics and engineering*, pp 71-159, *SEMA SIMAI Springer Ser.*, 9, Springer, [Cham], 2016.

[6] Photo credit: Q. Xu and H. Zhu. https://bit.ly/2GmsIJb. 2016.

Simulation: Memory effect on a viscoelastic polymer

Designed and produced by the presenter based on the team's MATLAB library. Numerical solution is computed by $\mathcal{P}_3$ elements on 6,144 tetrahedra with 1000 time-steps. Plotted on 124,416 boundary faces. Computation took 11 hours via 12 parallel MATLAB workers on UD HPC cluster.



Partially supported by NSF through grant DMS 1818867

{heruslu,fjsayas}@udel.edu